

Evolution of a Network of Vortex Loops in the Turbulent Superfluid Helium; Derivation of the Vinen Equation

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The evolution a network of vortex loops due to the fusion and breakdown in the turbulent superfluid helium is studied. We perform investigation on the base of the "rate equation" for the distribution function $n(l)$ of number of loops in space of their length l . There are two mechanisms for change of quantity $n(l)$. Firstly, the function changes due to deterministic process of mutual friction, when the length grows or decreases depending on orientation. Secondly, the change of $n(l)$ occurs due to random events when the loop crosses itself breaking down into two daughter or two loops collide merging into one larger loop. Accordingly the "rate equation" includes the "collision" term collecting random processes of fusion and breakdown and the deterministic term. Assuming, further, that processes of random colliding are fastest we are in position to study more slow processes related to deterministic term. In this way we study the evolution of full length of vortex loops per unit volume-so called vortex line density $\mathcal{L}(t)$. It is shown this evolution to obey the famous Vinen equation. In conclusion we discuss properties of the Vinen equation from the point of view of the developed approach. PACS numbers: 67.40.Vs 98.80.Cq 7.37.+q

1. INTRODUCTION AND SCIENTIFIC BACKGROUND

In spite of very long history the theory superfluid turbulence is very far from the more or less completeness (^{1,2,3}). The reason to this is an incredible complexity of this problem. Indeed we have to deal with a set of objects (vortex loops) which do not have a fixed number of elements, they can split

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and merge in processes of reconnection. Thus, some analog of the secondary quantization method is required with the difference that the objects (vortex loops) themselves possess an infinite number of degrees of freedom with very involved dynamics. Clearly this problem can hardly be resolved in the nearest future. As for direct numerical simulations, which remain the main source of information about this process^{4, 5, 6, 7} there are also many problems. It is clear that any progress can be achieved only if one can essentially reduce the number of degree of freedom of each of the loops. One way elaborated in context of lambda-transition⁽⁸⁾ is to imagine vortex loops as a set of rings of different sizes and to take their radius as the only degrees of freedom. In paper of author (see⁹) there is offered another way - to think of vortex loops as randomly walking chains. The physical ground for this supposition is the following one. During evolution the loops undergo huge amount of collisions with other loops (or with itself) with consequent reconnection. As a result each of the loops consists of many uncorrelated parts of mean size ξ_0 so called elementary step. A bit more precisely an average loop consists of smoothly (but randomly) connected arcs of mean radius ξ_0 . This fact allows to consider vortex loop as a random walk, and to describe its statistics with help of some generalized Wiener distribution. Parameters of this probability distribution functional can be regarded as degrees of freedom of vortex loop. This conception allowed to fulfil a series of studies on the vortex tangle properties, e. g. to calculate its momenta, energy and spectrum of energy. In the present work we demonstrate how this point conception allows to derive analytically the famous Vinen equation for evolution of the vortex line density $\mathcal{L}(t)$.

2. KINETIC EQUATION

Following the work¹⁰ we introduce the distribution function $n(l, t)$ of the density of a loop in the "space" of their lengths. It is defined as the number of loops (per unit volume) with lengths lying between l and $l + dl$. There are two mechanisms for change of $n(l, t)$. One of them is the deterministic process. In superfluid helium quantity $n(l, t)$ changes due to interaction of vortices with the normal component. This processes is more or less clear and can be modeled by relatively simple equations of motion. Other reasons for change of quantity $n(l, t)$ are the random reconnection processes. We discriminate two types of the reconnection processes, namely the fusion of two loops into the larger single loop and the breakdown of single loop into two daughter loops. The kinetics of the vortex tangle is affected by the intensity of the introduced processes. The intensity of the first process is characterized by the rate $A(l_1, l_2, l)$ (number of events per unit time and per

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unit volume) of collision of two loops with lengths l_1 and l_2 and forming the loop of length $l = l_1 + l_2$. The intensity of the second process is characterized by the rate of self-intersection $B(l, l_1, l_2)$ of the loop with length l into two daughter loops with lengths l_1 and l_2 . In view of what has been exposed above we can directly write out the master "kinetic" equation for rate of change the function $n(l, t)$.

$$\begin{aligned}
\frac{\partial n(l, t)}{\partial t} + \frac{\partial n(l, t)}{\partial l} \frac{\partial l}{\partial t} = & \int \int A(l_1, l_2, l) n(l_1) n(l_2) \delta(l - l_1 - l_2) dl_1 dl_2 \\
& - \int \int A(l_1, l, l_2) n(l) n(l_1) \delta(l_2 - l_1 - l) dl_1 dl_2 \\
& - \int \int A(l_2, l, l_1) n(l) n(l_2) \delta(l_1 - l_2 - l) dl_1 dl_2 \\
& - \int \int B(l_1, l_2, l) n(l) \delta(l - l_1 - l_2) dl_1 dl_2 \\
& + \int \int B(l, l_2, l_1) n(l_1) \delta(l_1 - l - l_2) dl_1 dl_2 \\
& + \int \int B(l, l_1, l_2) n(l_1) \delta(l_2 - l - l_1) dl_1 dl_2 .
\end{aligned} \tag{1}$$

Coefficients A and B are calculated in¹¹

$$A(l_1, l_2, l) = b_m V_l l_1 l_2, \quad B(l_1, l_2, l) = b_s V_l l (\xi_0 l_1)^{-3/2}.$$

Here b_m and b_s are some constants of order of unity. In paper¹¹ on vortex loops in superfluid turbulent HeII there was offered $b_m \approx 1/3$, $b_s \approx 0.0164772$. Quantity V_l is the characteristic velocity of lines by order magnitude equal κ/ξ_0 (κ is the quantum of circulation). The Brownian random walk approach fails for scales near ξ_0 , therefore usually this value appears as a low cut-off. In paper¹² there was shown that the collision term alone leads to steady solution expressed by formula

$$n(l) = C l^{-5/2}. \tag{2}$$

Solution (2) is nonequilibrium solution with the l -independent flux P of the length density $L = l n(l)$, which is the length accumulated in loops of length lying between l and $l + dl$. Term "flux" here means just the redistribution of length among the loops due to reconnections. Quantity P is equal

$$P = 6.2775 C^2 b_m \kappa / \xi_0 - 2.7725 C b_s \kappa \xi_0^{-5/2}. \tag{3}$$

The master equation (1) is the base for study of superfluid turbulence in the frame of conception of randomly walking chains. The aim of our present paper is to demonstrate that the vortex line density, total length per unit volume defined as $\mathcal{L}(t) = \int n(l) l dl$ evolves in accordance with the famous Vinen equation.²

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3. VINEN EQUATION

To show it let us multiply the kinetic equation (1) by l and integrate over all sizes

$$\frac{d\mathcal{L}(t)}{dt} = \int \frac{\partial n(l, t)}{\partial t} l dl = - \int \frac{\partial n(l, t)}{\partial l} \frac{\partial l}{\partial t} l dl - P_{net}. \quad (4)$$

Here the quantity P_{net} is the net "flux" of length in l -space, which is absolute value of flux P expressed by relation (3). This rule is imposed because flux P always carries away the vortex line density \mathcal{L} from the system, and different signs refers to direction of the cascade. First we treat the deterministic term in equation (4). Due to enormous number of reconnection it is quite natural to suppose that "equilibrium" solution (2) is reached much faster than the slow deterministic change of function $n(l)$. Then we can use this solution to evaluate deterministic term. We will calculate the rate of change of length of each loop on the base of the motion equation of the line in local approach (see e.g.⁴)

$$\mathbf{v}_l = \beta \mathbf{s}' \times \mathbf{s}'' + \alpha \mathbf{s}' \times (\mathbf{V}_{ns} - \beta \mathbf{s}' \times \mathbf{s}'') \quad (5)$$

Here \mathbf{s}' and \mathbf{s}'' are the first and second derivatives from position of line $\mathbf{s}(\xi)$ with respect to label variable ξ , which coincides here with the arclength. to calculate, $\mathbf{v}_l(\xi)$ is velocity of the line element. To calculate $\partial l / \partial t$, we use the relation for the rate of change of length $\partial \delta l / \partial t$ for some arbitrary element with length δl (see.g.⁴). Assuming for a while that the label variable ξ is not exactly the arclength, we have $\delta l = |\mathbf{s}'| \delta \xi$. Then the following chain of relations takes place.

$$\frac{\partial \delta l}{\partial t} = \frac{\partial |\mathbf{s}'| \delta \xi}{\partial t} = \frac{|\mathbf{s}'|}{|\mathbf{s}'|} \frac{\partial |\mathbf{s}'| \delta \xi}{\partial t} = \frac{\mathbf{s}'}{|\mathbf{s}'|} \frac{\partial \mathbf{s}' \delta \xi}{\partial t} = \mathbf{s}' \mathbf{v}_l' \delta \xi$$

On the last stage we return to $|\mathbf{s}'| = 1$. Differentiating (5) and multiplying by \mathbf{s}' we have after little algebra

$$\frac{\partial \delta l}{\partial t} = (\alpha (\mathbf{s}' \times \mathbf{s}'') \mathbf{V}_{ns} - \alpha \beta (\mathbf{s}' \times \mathbf{s}'')^2) \delta \xi \quad (6)$$

The next step is to average expression (6) over all possible configurations of vortex loops. We do it with use of the Gaussian model of the vortex tangle elaborated earlier by author.⁹ In accordance with this model

$$\langle \mathbf{s}' \times \mathbf{s}'' \rangle = \frac{I_l}{\sqrt{2} c_2 \xi_0} \frac{\mathbf{V}_{ns}}{|\mathbf{V}_{ns}|}, \quad \langle (\mathbf{s}' \times \mathbf{s}'')^2 \rangle = \langle (\mathbf{s}'')^2 \rangle = \frac{1}{2 \xi_0^2}. \quad (7)$$

Quantity c_2 is the structure constant introduced by Schwarz⁴). It follows from (6) and from (7) that $\partial \delta l / \partial t \propto l$. Substituting (7) into (averaged equation (6)) and then into (4) and integrating by part we get the contribution

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into $d\mathcal{L}(t)/dt$ from the deterministic term.

$$\left(\alpha \frac{I_l |\mathbf{V}_{ns}|}{\sqrt{2} c_2 \xi_0} - \alpha \beta \frac{1}{2 \xi_0^2}\right) \int \frac{\partial n(l, t)}{\partial l} l^2 dl = -\alpha \frac{2 I_l |\mathbf{V}_{ns}|}{\sqrt{2} c_2 \xi_0} \mathcal{L} + \frac{\alpha \beta}{\xi_0^2} \mathcal{L}. \quad (8)$$

Now we have to treat the "flux" term in equation (4). Unlike paper¹² where the constant C in relation (2) was obtained from condition $P = 0$, here we obtain C from the normalization condition $\mathcal{L}(t) = \int n(l) l dl = C \int l^{-3/2} dl = 2C/\xi_0$ (We recall that ξ_0 is a low cut-off of the whole approach and integral diverges on low limit). Furthermore we consider consequently the collision and reconnection events to put the system in equilibrium (with respect to solution (2)) state much faster than the slow deterministic processes. This implies that parameters ξ_0 and I_l , the so called structure constants of the vortex tangle, have a time to adjust to their equilibrium values. This assumption is widely adopted and it was confirmed in numerical simulations in.¹³ In particular the mean radius of curvature (playing a role of elementary step in our approach) is related to the vortex line density as $\xi_0^2 = 1/2 c_2^2 \mathcal{L}$ (see^{4,9}). By use of the said above we rewrite expression for flux (3) in form $P = C_F \kappa \mathcal{L}^2$, where the temperature constant C_F is

$$C_F = (1.5694 b_m - 2.7726 c_2^2 b_s). \quad (9)$$

We named this constant in honor of Feynman who was the first person to discuss evolution of vortex line density due to the reconnection processes. We would like to recall that he supposed decay of the vortex tangle due to the cascade like breakdown of vortex loops with further disappearance of them on very small scales. Relation (9) shows that there is possible the inverse cascade, which corresponds to the cascade-like fusion of vortex loops. Unfortunately our approach has too approximate character to do any strong quantitative conclusion. It is clear, however, that for low temperatures, where the vortex tangle is more kinky, correspondingly c_2 is large, the quantity C_F is negative. This corresponds to the direct cascade in region of very small loops. On the contrary for high temperature lines are smoother, c_2 is small, and C_F is positive, which implies that there is inverse cascade with formation of large loops. If we for instance adopt values for b_m and b_s and use for c_2^2 values offered by Schwarz (see⁴), then we get $C_F \approx -0.01$ for the temperature 1.07 K and $C_F \approx 0.44$ K for the temperature 2.01 K. Collecting contribution into $d\mathcal{L}(t)/dt$ (4) from both the deterministic and collision processes, and taking into account that $P_{net} = |C_F| \kappa \mathcal{L}^2$ we finally have

$$\frac{d\mathcal{L}(t)}{dt} = \frac{5}{2} \alpha I_l |\mathbf{V}_{ns}| \mathcal{L}^{\frac{3}{2}} - \frac{5}{2} \alpha \beta c_2^2 \mathcal{L}^2 - |C_F| \kappa \mathcal{L}^2. \quad (10)$$

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Thus, starting with kinetics of a network of vortex loops, we get the famous Vinen equation.² Let us discuss meaning of various terms entering this equation. The first, generating term in the rhs of the Vinen equation describes the grows of the vortex tangle due to mutual friction. The second term also connected to mutual friction, however this term is responsible for decrease of the vortex line density. This point of view coincides with ideas by Schwarz⁴ who obtained the deterministic contribution into $d\mathcal{L}(t)/dt$ using a bit different approach. The third term in the RHS of (10) is related to random collisions of vortex loops. It describes decrease of the vortex line density due to the flux of length carrying away the length from the system. Depending on interplay between coefficients b_m and b_s and parameters c_2 the flux can be either positive or negative. Negative flux appears when break down of loops prevails and cascade-like process of generation of smaller and smaller loops forms. There exists a number of mechanisms of disappearance of rings on very small scales. It can be e.g. acoustic radiation, collapse of lines, Kelvin waves etc. Thus, in this case the situation is fully coincides with the scenario proposed by Feynman.¹ The case when the flux is positive is less clear. Positive flux implies the cascade-like process of generation of larger and larger loops. Unlike previous case of negative flux, there is no apparent mechanism for disappearance of very large loops. There are possible the following possibilities. The first one is that very large loops, whose $3D$ size is comparable with size of the volume (note that length of these loop is much larger because of the random walk structure) can annihilate fully or partly on boundaries. Then the picture reminds the Feynman scenario but with the inverse flux. The second possibility is that parts of large loops is pinned on the walls and the whole approach representing the line to have the random walk structure fails. Finally it should not be ruled out, that a state with lines pinned on the walls and stretching from wall to wall is just a degenerate state of the vortex tangle. Some of numerical investigators^{4,6} report on this situation.

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